Extensions of the auxiliary field method to solve Schrödinger equations

Bernard Silvestre-Brac[1]**, Claude Semay**[2] **and Fabien Buisseret**[2]

[1] LPSC Université Joseph Fourier, Grenoble 1, CNRS/IN2P3, Institut Polytechnique
de Grenoble, Avenue des Martyrs 53, F-38026 Grenoble-Cedex, France
[2] Groupe de Physique Nucléaire Théorique, Université de Mons-Hainaut, Académie universitaire
Wallonie-Bruxelles, Place du Parc 20, B-7000 Mons, Belgium

E-mail: silvestre@lpsc.in2p3.fr, claude.semay@umh.ac.be and fabien.buisseret@umh.ac.be

Abstract

It has recently been shown that the auxiliary field method is an interesting tool to compute approximate analytical solutions of the Schrödinger equation. This technique can generate the spectrum associated with an arbitrary potential $V(r)$ starting from the analytically known spectrum of a particular potential $P(r)$. In the present work, general important properties of the auxiliary field method are proved, such as scaling laws and independence of the results on the choice of $P(r)$. The method is extended in order to find accurate analytical energy formulae for radial potentials of the form $aP(r) + V(r)$, and several explicit examples are studied. Connections existing between the perturbation theory and the auxiliary field method are also discussed.

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1. Introduction

Auxiliary fields, also known as einbein fields, have been known about for a long time in quantum field theory. Initially they were introduced to remove the cumbersome square roots appearing in relativistic theories. As an example in string field theory, let us cite the Nambu-Goto Lagrangian which is transformed into the Polyakov Lagrangian [1]. They are also of common use in many other fields of physics, such as supersymmetric field theories [2] and hadronic physics [3]. A particular use of the auxiliary fields is the transformation of a semi-relativistic kinetic energy term ($\sqrt{p^2 + m^2}$) appearing in Salpeter-type equations into an apparently non-relativistic one ($p^2/(2\mu)$) leading to a simpler Schrödinger-like equation [4]. Another interesting approach involving auxiliary fields is the transformation of a problem containing a linear confining term into a new one containing a harmonic oscillator potential which can lead to analytical expressions [5].

Recently, it was realized [6] that auxiliary fields can be used as a tool to obtain analytical approximate expressions for the eigenvalues and the eigenstates of a Schrödinger equation. In this work, hereafter labelled SSB, we proposed a systematic method, called the auxiliary field method (AFM), which gives approximate expressions for the eigenenergies of a non-relativistic two-body system interacting through any local and central potential $V(r)$. For special forms of this potential, an analytical expression is available. In SSB we studied in particular power-law and logarithmic potentials and proposed new energy formulae which are much more accurate than those found in the literature up to now.

The search for analytical solutions of the Schrödinger equation is very interesting and the subject of important investigations. In fact, only very few potentials give rise to an analytical expression for the eigenenergies valid for any values of the radial quantum number n and orbital quantum number l. The most famous are the harmonic oscillator $V(r) = \frac{1}{2}\omega r^2$ and the Coulomb potential $V(r) = -\kappa/r$.

Other potentials have analytical solutions but only for *S*-waves (or in a one-dimensional case). This is in particular the case for the Morse potential, the Hulthen potential, the Hylleraas potential or the Eckart potential. There also exist potentials which are solvable but for particular values of their parameters. Among them, the Kratzer potential $V(r) = 2D\left(\frac{1}{2}a^2/r^2 - b/r\right)$ is famous and analytically solvable in the case $b = a$ [7].

But even if an exact analytical expression is not available, it may be very interesting to have an approximate analytical expression at our disposal. In addition to a possible benchmark for numerical calculations, it exhibits the explicit dependence of the energies as a function of the various parameters and of the quantum numbers. Moreover, an analytical expression is always much less time consuming than the corresponding numerical resolution, and is thus of very great help in the case of a search for a set of parameters for a potential relying on a chi-square best fit.

Several methods have been invoked to find approximate analytical solutions: WKB method, semi-classical treatment, variational methods, perturbation theory, etc. In SSB we showed that the AFM is especially well suited to pursue this goal and presents very pleasant features. The aim of this work, which extends the results obtained in SSB, is essentially twofold:

- to demonstrate a number of very general and interesting properties of the AFM, in particular the connections with the perturbation theory;
- to obtain approximate analytical expressions for a wide class of potentials, some of them being very important and of common use in several domains of physics.

The paper is organized as follows. In the second section, we demonstrate a number of very general properties concerning the AFM. The third section is devoted to the application of the general theory to very specific but important potentials. The fourth section deals with a detailed comparison of our analytical results and the corresponding numerical values. Some conclusions are drawn in the last section.

2. General properties

2.1. Principle of the method

Our goal is the search for approximate analytical expressions of the eigenvalues for a two-body non-relativistic Hamiltonian

$$H = \frac{\boldsymbol{p}^2}{2m} + V(r), \tag{1}$$

where m is the reduced mass, r is the relative distance between particles, $\boldsymbol{p}$ is the conjugate momentum associated with $\boldsymbol{r}$ and $V(r)$ is any central potential. An analytical expression for all the eigenvalues is known explicitly only for very specific potentials $P(r)$. The basic idea is to rely on such potentials. In other words, we assume that we are able to obtain an analytical expression $e(a)$ for the Schrödinger equation

$$\bar{H}(a)\,|a\rangle = \left[\frac{\boldsymbol{p}^2}{2m} + a\,P(r)\right]|a\rangle = e(a)|a\rangle, \tag{2}$$

in which, at this stage, a is a real parameter.

Our method needs the introduction of an auxiliary field ν, which is *a priori* an operator. We recall here the principle of the method. It consists in four steps:

(i) We calculate the function (the prime denotes the derivative with respect to r)

$$K(r) = \frac{V'(r)}{P'(r)} \tag{3}$$

and denote by $\hat{\nu}$ the value of the auxiliary field which coincides with this function,

$$\hat{\nu} = K(r). \tag{4}$$

(ii) We denote by $J = K^{-1}$ the inverse function of K. Thus, one has

$$r = J(\hat{\nu}). \tag{5}$$

Since both $V(r)$ and $P(r)$ do exhibit an analytical form, the same property holds for $K(r)$. But it is by no means sure that $J(\hat{\nu})$ can be expressed analytically. The existence of an analytical expression for J is a necessary condition for our method to obtain analytical expressions as a final result.

(iii) We build a new Hamiltonian $\tilde{H}$ which depends on the auxiliary field ν through the following expression:

$$\tilde{H}(\nu) = \bar{H}(\nu) + g(\nu), \tag{6}$$

where $\bar{H}$ is defined by (2) and where the function $g(\nu)$ is given explicitly by

$$g(\nu) = V(J(\nu)) - \nu P(J(\nu)). \tag{7}$$

This function $g(\nu)$ makes a bridge between the potential $P(r)$ for which an analytical expression is known and the potential $V(r)$ for which an analytical expression is *a priori* not known. The very important property is that $\hat{\nu}$, coming from (4), cancels the variation of the new Hamiltonian, i.e. $\delta\tilde{H}(\nu)/\delta\nu|_{\nu=\hat{\nu}} = 0$. Moreover, one has the additional crucial identity $\tilde{H}(\hat{\nu}) = H$, as defined by (1).

(iv) Considering now ν no longer as an operator but as a pure number and taking into account (2), the eigenvalues of $\tilde{H}(\nu)$ are

$$E(\nu) = e(\nu) + g(\nu), \tag{8}$$

where $e(\nu)$ are the eigenvalues of $\bar{H}$. Then, we determine the value ν_0 that minimizes $E(\nu)$: $\partial E(\nu)/\partial\nu|_{\nu_0} = 0$. We propose to consider $E(\nu_0)$ as the approximate eigenvalues of the Hamiltonian H. In SSB, we presented a bound to test the accuracy of the method. In order to obtain an analytical expression for these eigenvalues, we must fulfil a second necessary condition: to be able to determine ν_0 and, then, $E(\nu_0)$ in an analytical way.

This method is completely general and *a priori* valid for any potential $V(r)$. Nevertheless, in order to get analytical expressions we must fulfil, as we saw, two conditions: (i) first, to be able to invert relation (4) in order to have access to the function $J(\hat{v})$ defined by (5) and (ii) second, to be able to determine v_0 and to calculate the corresponding value $E(v_0)$ in an analytical way.

An idea for obtaining analytical expressions of the eigenenergies for an arbitrary potential is the following. We start with a potential $P(r) = P^{[0]}(r)$ for which the energies of the corresponding Hamiltonian $H^{[0]}$ are exactly known. We then proceed as above to find approximate solutions for the eigenenergies of a Hamiltonian $H^{[1]}$ in which the potential is at present $V(r) = P^{[1]}(r)$. In general, a large class of potentials can be treated in that way. Moreover, by comparison with accurate numerical results, we can even refine the expressions in order to be very close to the exact solution.

Considering now these approximate expressions as the exact ones, we apply once more the AFM with $P(r) = P^{[1]}(r)$ to obtain approximate solutions for the eigenenergies of a Hamiltonian $H^{[2]}$ in which the potential is at present $V(r) = P^{[2]}(r)$. Even if analytical solutions for Hamiltonian $H^{[2]}$ were not attainable directly with $P(r) = P^{[0]}(r)$, it may occur that they indeed are with $P(r) = P^{[1]}(r)$. Pursuing recursively such a procedure, one can imagine to get analytical solutions even for complicated potentials. Presumably, the quality of the analytical expressions deteriorates with the order of the recursion.

2.2. Expression of approximate energies

Very rare are the potentials $P(r)$ for which an analytical solution is known for all radial n and orbital l quantum numbers. Among them, the harmonic oscillator (ho) $P(r) = r^2$ and the Coulomb (C) $P(r) = -1/r$ potentials are widely used. Taking benefit of this opportunity, the class of power-law potentials (pl) has been studied in SSB. Let us consider $P^{(\lambda)}(r) = \text{sgn}(\lambda)r^\lambda$ with $\text{sgn}(\lambda) = \lambda/|\lambda|$ and $\lambda \neq 0$. It has been shown in SSB that the eigenvalues of the Hamiltonian

$$H_\lambda^{(\text{pl})}(a) = \frac{p^2}{2m} + a\,\text{sgn}(\lambda)r^\lambda \tag{9}$$

can be written under the form

$$e_\lambda^{(\text{pl})}(a) = \frac{2+\lambda}{2\lambda}(a|\lambda|)^{2/(\lambda+2)}\left(\frac{N_\lambda^2}{m}\right)^{\lambda/(\lambda+2)}, \tag{10}$$

where N_λ depends on n and l quantum numbers, as well as λ. This formula gives the exact result for the two important situations:

- the harmonic oscillator potential since in this case $N^{(\text{ho})} = N_{\lambda=2} = 2n + l + 3/2$;
- the Coulomb potential for which $N^{(\text{C})} = N_{\lambda=-1} = n + l + 1$.

It has been shown in SSB that, for any physical values of λ ($\lambda > -2$), a good form for N_λ is given by

$$N_\lambda = b(\lambda)n + l + c(\lambda). \tag{11}$$

Some functions $b(\lambda)$ and $c(\lambda)$ were proposed in SSB in order to give an approximation as precise as 10^{-3} for the most interesting (the lowest) values of the quantum numbers n and l.

Considering the eigenenergies (10) and (11) as the 'exact' ones for Hamiltonian (9), one can apply the AFM to get the eigenenergies for Hamiltonian (1). Using the recipe given in section 2.1 with $P(r) = P^{(\lambda)}(r)$, we find the following results:

$$E(v_0) = \frac{|\lambda|}{2}v_0 J(v_0)^\lambda + V(J(v_0)), \tag{12}$$

the optimal value ν_0 being determined from the equation

$$|\lambda|\nu_0 J(\nu_0)^{\lambda+2} = \frac{N_\lambda^2}{m} = Y_\lambda \tag{13}$$

and the function $J(\nu)$ coming from the relation

$$|\lambda|\nu J(\nu)^{\lambda-1} = V'(J(\nu)). \tag{14}$$

Let us emphasize that $J(\nu)$ depends only on the potential $V(r)$ and not on the particular eigenstate we are interested in. The value ν_0 depends both on the potential (through the J function) and on the state under consideration (through the N_λ quantity). It is important to stress that the expression resulting from (12) suffers from two approximations:

- the AFM is based on the replacement of an operator ν by an optimal value ν_0; this approximation was discussed in detail in SSB and an estimation of the error was given;
- setting for N_λ the value (11) is also an approximation whose quality was discussed extensively in SSB.

In the case of a harmonic oscillator $\lambda = 2$ or a Coulomb potential $\lambda = -1$, only the first type of approximation remains.

The application to the two solvable potentials is immediate:

- For the Coulomb potential ($\lambda = -1$), one deduces

$$E^{(C)}(\mu_0) = \frac{\mu_0}{2J(\mu_0)} + V(J(\mu_0)) = \frac{(N^{(C)})^2}{2mJ(\mu_0)^2} + V(J(\mu_0)). \tag{15}$$

The function $J(\mu)$ is determined by the condition

$$V'(J(\mu))J(\mu)^2 = \mu, \tag{16}$$

whereas the value μ_0 is calculated from the transcendental equation

$$\mu_0 J(\mu_0) = \frac{(N^{(C)})^2}{m}. \tag{17}$$

- For the harmonic oscillator ($\lambda = 2$), one has similarly

$$E^{(ho)}(\nu_0) = \nu_0 I(\nu_0)^2 + V(I(\nu_0)) = \frac{(N^{(ho)})^2}{2mI(\nu_0)^2} + V(I(\nu_0)). \tag{18}$$

The function $I(\nu)$ is determined by the condition

$$V'(I(\nu)) = 2\nu I(\nu), \tag{19}$$

whereas the value ν_0 is calculated from the transcendental equation

$$\nu_0 I(\nu_0)^4 = \frac{(N^{(ho)})^2}{2m}. \tag{20}$$

The comparison of the expression for the energies in both cases (15) and (18) clearly shows that there must exist a link between them. This aspect is considered in the following section and indeed we will prove a very interesting property.

2.3. Switching from $P^{(\lambda)}(r)$ to $P^{(\eta)}(r)$

In the preceding section, we derived the approximation E_λ that can be obtained for the eigenvalues of Hamiltonian (1) applying the AFM with the starting potential $P^{(\lambda)}(r)$. Let us assume that, instead of the starting potential $P^{(\lambda)}(r)$, we are interested by another starting potential $P^{(\eta)}(r)$. Formulae (12)–(14) apply as well, changing λ into η. In particular, one determines a function $I(\mu)$ depending on the auxiliary field μ, an optimal value μ_0 depending on a Y_η quantity and the resulting energy $E_\eta(\mu_0)$. In this last approach, let us introduce a new field ν by the change of variable

$$\nu = \nu(\mu) = \frac{|\eta|}{|\lambda|} \mu I(\mu)^{\eta-\lambda}. \tag{21}$$

Using the definition of $I(\mu)$ as a function of the potential V, this new variable can be defined as well as

$$\nu = \nu(\mu) = \frac{1}{|\lambda|} I(\mu)^{1-\lambda} V'(I(\mu)). \tag{22}$$

Using the definition of the J function (14), it is easy to show the relationship

$$I(\mu) = J(\nu(\mu)). \tag{23}$$

Defining $\nu_0 = \nu(\mu_0)$, a simple calculation shows that

$$|\lambda| \nu_0 J(\nu_0)^{\lambda+2} = \frac{N_\eta^2}{m} = Y_\eta. \tag{24}$$

This is exactly the expression (13) but in which the quantity Y_λ has been replaced by Y_η.

To achieve the demonstration, let us introduce this value $\nu_0(Y_\eta)$ in the expression of $E_\lambda(\nu_0)$. Using the link between ν_0 and μ_0, it is easy to show that

$$E_\lambda(\nu_0(Y_\eta)) = \frac{|\eta|}{2} \mu_0 I(\mu_0)^\eta + V(I(\mu_0)) = E_\eta(\nu_0(Y_\eta)). \tag{25}$$

In the expression derived from the case $P^{(\lambda)}(r)$, it is sufficient to change the value N_λ by N_η to obtain the expression derived from the case $P^{(\eta)}(r)$. We end up with the very important conclusion that can be stated as a theorem:

If, in the expression $E(N_\lambda)$ of the approximate energies resulting from the AFM with $P^{(\lambda)}(r)$, one makes the substitution $N_\lambda \to N_\eta$ (so that $E(N_\lambda) \to E(N_\eta)$ with the same functional form for E), one obtains the approximate eigenenergies resulting from the AFM with $P^{(\eta)}(r)$.

In a sense, as long as we use a power-law potential $P^{(\lambda)}(r)$ as starting potential, there is a universality of the approximate AFM expression of the eigenvalue, depending only on the potential $V(r)$. The only reminiscence of the particular chosen potential $P^{(\lambda)}(r)$ is the expression of N_λ, as given by (11). This result holds whatever the form chosen for the potential $V(r)$, even if we are unable to obtain analytical expressions for one case or the other or both.

This property was emphasized in SSB for the particular case of a power-law potential $V(r) = r^\lambda$ switching from the harmonic oscillator ($\lambda = 2$) to the Coulomb potential ($\lambda = -1$). We proved here that it is in fact totally general. It is probably related to a well-known property in classical mechanics: one can pass from the motion of a harmonic oscillator to the Kepler motion by a canonical transformation.

2.4. Scaling laws

Scaling laws represent an important property for non-relativistic Schrödinger equations. They allow us to give the expression for the eigenenergies (and wavefunctions) of the most general

equation in terms of the corresponding eigenenergies (and wavefunctions) of a reduced equation which is much simpler to solve.

Let us recall briefly the scaling law for the energy. Let $E(m, G, a)$ be the eigenvalues of a Schrödinger equation corresponding to a system of reduced mass m subject to a potential of intensity G and characteristic inverse length a. The scaling law gives the relationship between $E(m, G, a)$ and $E(m', G', a')$. Let us start from the corresponding Schrödinger equations

$$\left[-\frac{1}{2m} \Delta_r + G V(ar) - E(m, G, a) \right] \Psi(\boldsymbol{r}) = 0, \tag{26}$$

$$\left[-\frac{1}{2m'} \Delta_r + G' V(a'r) - E(m', G', a') \right] \Psi'(\boldsymbol{r}) = 0. \tag{27}$$

The important point is that it is the same function $V(x)$ which appears in both equations. In (27), let us make the change of variables $\boldsymbol{r} = \alpha\boldsymbol{x}$ and multiply it by χ. Now, we choose the arbitrary parameters α and χ in order to fulfil the conditions $\chi/(m'\alpha^2) = 1/m$ and $\alpha a' = a$. In other words, we impose the following values:

$$\alpha = \frac{a}{a'}, \qquad \chi = \frac{m'}{m} \left(\frac{a}{a'} \right)^2 . \tag{28}$$

With these values, (27) can be recast into the form

$$\left[-\frac{1}{2m} \Delta_x + G'\frac{m'}{m} \left(\frac{a}{a'} \right)^2 V(ax) - \frac{m'}{m} \left(\frac{a}{a'} \right)^2 E(m', G', a') \right] \Psi'\left(\frac{a}{a'} \boldsymbol{x} \right) = 0. \tag{29}$$

Equation (26) can be recovered, provided one makes the identification $G = G'(m'/m)(a/a')^2$ and a similar relation for the energies.

The scaling law is thus expressed in its most general form as

$$E(m, G, a) = \frac{m'}{m} \left(\frac{a}{a'} \right)^2 E\left(m', G' = G\frac{m}{m'} \left(\frac{a'}{a} \right)^2, a' \right). \tag{30}$$

In fact, it is always possible to define the function $V(x)$ so that $a' = 1$. In what follows, and without loss of generality, we will apply the scaling law for energies under the form

$$E(m, G, a) = \frac{m'a^2}{m} E\left(m', G' = \frac{mG}{m'a^2}, 1 \right). \tag{31}$$

This equality is very powerful since it is valid for the *exact eigenvalues* of a non-relativistic Schrödinger equation based on an *arbitrary central potential*. It allows us to express the energy in terms of a dimensionless quantity and some dimensioned factors, as we will see below.

The purpose of this section is to prove that the scaling law for energy, as expressed by (31) still holds for the approximate expressions derived from a treatment based on auxiliary fields. It was observed in SSB but no proof was given. The only assumption is that the function P is homogeneous, that is we impose the property $P(ar) = a^p P(r)$ which also implies $P'(ar) = a^{p-1} P'(r)$. Let us denote here by $M(m, G)$ the eigenvalues of the Hamiltonian

$$\bar{H}(G) = \frac{p^2}{2m} + G P(r). \tag{32}$$

In our treatment, the analytical expression for $M(m, G)$ is supposed to be known. We are searching for the approximate expression of the eigenvalues of Hamiltonian (1) with $V(r) = Gv(ar)$. We call J the inverse function of v'/P'; consequently we have the property $v'(J(X)) = X P'(J(X))$.

　　　Let us apply the recipe described in section 2.1. The function $K(r) = Gav'(ar)/P'(r) = Ga^p v'(ar)/P'(ar)$ must be identified with the auxiliary field $\hat{v}$ and its inversion provides $r(\hat{v})$ with

$$ar(\hat{v}) = J(\hat{v}/(Ga^p)). \tag{33}$$

The auxiliary Hamiltonian to be considered reads

$$\tilde{H}(v) = \frac{p^2}{2m} + vP(r) + G[v(J(\mu)) - \mu P(J(\mu))] \tag{34}$$

with

$$\mu(v) = \frac{v}{Ga^p}. \tag{35}$$

The corresponding eigenenergies are thus

$$E(m, G, a; v) = M(m, v) + G[v(J(\mu)) - \mu P(J(\mu))]. \tag{36}$$

The next step now is the determination of the value v_0 which minimizes this value of E. A simple calculation shows that the value of $\mu_0 = \mu(v_0)$ is explicitly given by

$$a^p M'(m, Ga^p \mu_0) = P(J(\mu_0)), \tag{37}$$

where $M'(m, v) = \partial M(m, v)/\partial v$.

　　　Lastly, the energy $E(m, G, a)$ we are looking for is just the value $E(m, G, a; v_0)$. The final expression is therefore

$$E(m, G, a) = M(m, Ga^p \mu_0) + G[v(J(\mu_0)) - \mu_0 P(J(\mu_0))]. \tag{38}$$

Since (31) is valid whatever the potential, it is in particular valid for the energies M. One deduces the following relations:

$$M(m', mG\mu_0'/(m'a^2)) = (m/(m'a^2))M(m, Ga^p \mu_0') \tag{39}$$

and, after differentiation,

$$M'(m', mG\mu_0'/(m'a^2)) = a^p M'(m, Ga^p \mu_0'). \tag{40}$$

　　　Now let us consider the value

$$E(m', G' = (mG)/(m'a^2), 1) = M(m', mG\mu_0'/(m'a^2)) \tag{41}$$

$$+ (mG)/(m'a^2)[v(J(\mu_0')) - \mu_0' P(J(\mu_0'))]. \tag{42}$$

The value of μ_0' is obtained from (see (37) applied with $a = 1$)

$$M'(m', mG\mu_0'/(m'a^2)) = P(J(\mu_0')) \tag{43}$$

which, because of the property (40), can be transformed into

$$a^p M'(m, Ga^p \mu_0') = P(J(\mu_0')). \tag{44}$$

A comparison between (37) and (44) immediately implies that $\mu_0' = \mu_0$. Thanks to (39), this last identity inserted in (41) proves that

$$E(m, G, a) = \frac{m'a^2}{m} E\left(m', G' = \frac{mG}{m'a^2}, 1\right). \tag{45}$$

But this result is precisely what we expect from the scaling law (see (31)).

　　　We thus proved that scaling law for energy holds as well for the approximate expressions derived by the AFM. This result is almost general in the sense that it is valid whatever the potential under consideration, even if we are unable to obtain analytical solution. The only restriction is that the function $P(r)$ is homogeneous. This is of no consequence since the potential $P(r) = \text{sgn}(\lambda)r^\lambda$ is used in practice.

2.5. Extension of the method

Given a potential $V(r)$, the method for obtaining approximate solutions using auxiliary fields has been presented extensively in section 2.1. This method can be extended without difficulty for a Hamiltonian of type

$$H_a = \frac{p^2}{2m} + aP(r) + V(r). \tag{46}$$

One introduces an auxiliary field as before forgetting about the $aP(r)$ contribution. The first three steps of the algorithm remain unchanged. Thus the $\hat{v}$ field is the same, as is the same the function $g(v)$. The only difference arises in the expression (6) of $\tilde{H}$ and $\bar{H}$ where $vP(r)$ has to be replaced by $(a + v)P(r)$. As a consequence, the corresponding energy (8) has to be replaced by

$$E_a(v) = e(a + v) + g(v). \tag{47}$$

$E_a(v)$ is an eigenvalue of the Hamiltonian

$$\tilde{H}_a(v) = \bar{H}(a + v) + g(v), \tag{48}$$

where $\bar{H}$ is defined by (2) and where $e(a + v)$ is an eigenvalue of Hamiltonian $\bar{H}(a + v)$. An eigenstate of Hamiltonians $\tilde{H}_a$ and $\bar{H}(a + v)$ is denoted $|a + v\rangle$, and we have $e(a + v) = \langle a + v|\bar{H}(a + v)|a + v\rangle$. If v_0 is the value of v which minimizes (47), then we could expect that

$$E_a(v_0) = e(a + v_0) + g(v_0) \tag{49}$$

is a good approximation of E_a, an eigenvalue of Hamiltonian (46). It seems that (47) is very similar to (8). Nevertheless, the small difference is important, because, even if the determination of v_0 from (8) is technically easy, it may happen that the determination from (47) could be much more involved.

Using the Hellmann–Feynman theorem [8] as in SSB, it can be shown that

$$\langle a + v_0|P(r)|a + v_0\rangle = P(J(v_0)). \tag{50}$$

So, $J(v_0)$ is a kind of 'average point' for the potential $P(r)$. If functions $P(r)$ and $V(r)$ are not too different, one could expect that

$$\langle a + v_0|V(r)|a + v_0\rangle \approx V(J(v_0)), \tag{51}$$

and, in particular, that (see (4) and (5))

$$\langle a + v_0|\hat{v}|a + v_0\rangle \approx v_0. \tag{52}$$

Within this condition, the optimal value of the constant replacing the auxiliary field operator is close to the mean value of this operator, as mentioned in SSB. So, the AFM could actually be considered as a 'mean field approximation' with respect to a particular auxiliary field which is introduced to simplify the calculations. As is also shown in SSB, using (50), one obtains

$$E_a(v_0) - E_a \gtrsim V(J(v_0)) - \langle a + v_0|V(r)|a + v_0\rangle. \tag{53}$$

Except E_a, all quantities can be computed analytically in principle. So an estimation of the accuracy of the approximate eigenvalue $E_a(v_0)$ can be obtained.

It is very instructive to apply (47) for the potential $P(r) = P^{(\eta)}(r)$. In this case, the function $J(v)$ is unchanged and still given by (14) (with the obvious change $\lambda \to \eta$), while the new value of v_0 is given, instead of (13), by

$$|\eta|(a + v_0)J(v_0)^{\eta+2} = \frac{N_\eta^2}{m} \tag{54}$$

and the corresponding value of the energy by

$$E(\nu_0) = \text{sgn}(\eta)\frac{(2+\eta)a + \eta\nu_0}{2}J(\nu_0)^\eta + V(J(\nu_0))$$

$$= \text{sgn}(\eta)\frac{(2+\eta)a + \eta\nu_0}{2}\left(\frac{N_\eta^2}{m|\eta|(a+\nu_0)}\right)^{\eta/(2+\eta)} + V(J(\nu_0)). \qquad (55)$$

In this case, the virial theorem states that

$$e(a + \nu_0) = \frac{\lambda+2}{2}\langle a + \nu_0|(a+\nu_0)\text{sgn}(\lambda)r^\lambda|a+\nu_0\rangle, \qquad (56)$$

with $e(z)$ given by (10). It is then easy to verify explicitly the equality (50) with $J(\nu_0)$ given by (54). These relations also imply that

$$\langle a + \nu_0|r^\lambda|a+\nu_0\rangle = J(\nu_0)^\lambda. \qquad (57)$$

But, an outcome of (51) is $\langle a + \nu_0|r|a+\nu_0\rangle \approx J(\nu_0)$ with $J(\nu_0)$ given by (54). So this relation is equivalent to the approximation $\langle a + \nu_0|r^\lambda|a+\nu_0\rangle \approx \langle a + \nu_0|r|a+\nu_0\rangle^\lambda$.

With this in mind, a potential that is worthwhile to be studied is the sum of two power-law potentials, namely

$$V(r) = \text{sgn}(\eta)ar^\eta + \text{sgn}(\lambda)br^\lambda. \qquad (58)$$

The above considerations show that the $J(\nu)$ function is given by

$$J(\nu) = \left(\frac{|\eta|\nu}{|\lambda|b}\right)^{1/(\lambda-\eta)}, \qquad (59)$$

with the consequence that the optimal value ν_0 is extracted from the equation

$$(a + \nu_0)\nu_0^{(\eta+2)/(\lambda-\eta)} = \frac{N_\eta^2(|\lambda|b)^{(\eta+2)/(\lambda-\eta)}}{m|\eta|^{(\lambda+2)/(\lambda-\eta)}}. \qquad (60)$$

In general, one does not have an analytical expression for the root of such an equation. This is possible only for very specific values of the powers η and λ.

In this paper, we will study such a problem for the most favourable cases, where η is chosen to give an exact expression for the eigenvalues (in practice $\eta = 2$ and $\eta = -1$) and where λ is chosen in order to have an analytical root.

2.6. Relation between AFM and perturbation theory

Let us assume that $V(r) = \sigma v(r)$ with σ small enough so that $\sigma v(r) \ll aP(r)$. If σ is strictly 0, (46) and (48) show that ν, and hence ν_0, vanishes. Switching on the potential gives a non-vanishing but small value of ν_0. Indeed, $\hat{\nu} = \sigma v'(r)/P'(r)$, so we can expect that $\nu_0 \propto O(\sigma)$. Let us remark that $J(0)$ can have a finite value, as can be seen on (54) for the particular case $P(r) = P^{(\eta)}(r)$.

From results of the previous section, we can write

$$E_a(\nu) = e(a + \nu) + \sigma v(J(\nu)) - \nu P(J(\nu)). \qquad (61)$$

Using the definition (4), the condition $\partial E_a(\nu)/\partial\nu|_{\nu=\nu_0} = 0$ implies that

$$\left.\frac{\partial e(a+\nu)}{\partial\nu}\right|_{\nu=\nu_0} = e'(a+\nu_0) = P(J(\nu_0)). \qquad (62)$$

It is interesting to compare this relation with (50). Equation (61) turns then into

$$E_a(\nu_0) = e(a + \nu_0) - \nu_0 e'(a+\nu_0) + \sigma v(J(\nu_0)). \qquad (63)$$

Expanding $e(a + v_0)$ and $e'(a + v_0)$ in powers of σ and keeping only the terms $O(\sigma)$, we obtain

$$E_a(v_0) \approx e(a) + \sigma v(J(v_0)). \tag{64}$$

For small values of σ, the contribution of $\sigma v(r)$ can also be computed in perturbation. With the notations given above, we have then

$$E_{\text{pert.}} = \langle a|H_a|a\rangle = e(a) + \sigma \langle a|v(r)|a\rangle, \tag{65}$$

where $|a\rangle$ is an eigenstate of the Hamiltonian $\bar{H}(a)$. Since $|a + v_0\rangle$ and $|a\rangle$ differ only by terms $O(\sigma)$, we have (see (50))

$$P(J(v_0)) = \langle a + v_0|P(r)|a + v_0\rangle = \langle a|P(r)|a\rangle + O(\sigma). \tag{66}$$

If the condition (51) is fulfilled, we can also write

$$\langle a|v(r)|a\rangle \approx v(J(v_0)) + O(\sigma). \tag{67}$$

Using this result in (65), we see that $E_{\text{pert.}}$ and $E_a(v_0)$ differ only by terms $O(\sigma^2)$. Thus, the AFM and the perturbation theory give exactly the same results at first order provided $V(r)$ do not differ too strongly from $P(r)$. Let us remark that the perturbation method needs the computation of $\langle a|v(r)|a\rangle$. The AFM shows that this calculation can be replaced by the computation of $v(J(v_0))$ which could be simpler in some particular cases.

3. Application to special potentials

In SSB, we exploited the AFM for power-law potentials $V(r) = br^\lambda$. It was shown that an analytical expression for the energies exists for any value of λ (we mainly focused our attention on values of λ comprised between -1 and $+2$) and is given by (10). In this section, we take benefit of the remark of section 2.5 to study potentials of the form $aP(r)^{(\eta)} \pm br^\lambda$. As stated above, an analytical expression is not necessarily available. We will examine first in which cases we do have an analytical expression, and then we will investigate in more details some of them. In the following, it is assumed that $a > 0, b > 0$ and $-2 < \lambda \leqslant 2$.

3.1. Solvable potentials

Let us examine first the case $P(r) = r^2$. Application of the general method to this particular case leads to the following equation for the determination of v_0 (see (60)):

$$a \pm v_0 = X v_0^{\frac{4}{2-\lambda}}, \tag{68}$$

where $X = X(m, b, N^{(\text{ho})})$ is some function of the parameters whose expression does not matter for our purpose. This equation is obviously a transcendental equation for which an analytical solution does not exist automatically. The only cases for which we are sure that an analytical solution exists is when it can be transformed into a polynomial of degree less than or equal to 4. In order to investigate this condition, let us consider $4/(2 - \lambda) = p/q$ as a rational number (p and q are relatively prime). Calling $x = v_0^{1/q}$, (68) is recast as

$$a \pm x^q = X x^p. \tag{69}$$

All the solvable potentials should verify the conditions $1 \leqslant p \leqslant 4, 1 \leqslant q \leqslant 4$. An exhaustive research of all the solvable potentials leads to the following values of the power λ:

$$\lambda = -2, \ -1, \ -\tfrac{2}{3}, \ \tfrac{2}{3}, \ 1. \tag{70}$$

We will study in detail the cases:

- $\lambda = -2$ because it corresponds to a centrifugal term with a real parameter instead of the usual $l(l+1)$ term;
- $\lambda = -1$ because it corresponds to a simplified potential for hadronic systems with a short-range Coulomb potential and a quadratic confinement;
- $\lambda = 1$ because this anharmonic potential is sometimes used in molecular physics.

Now we investigate the case $P(r) = -1/r$. The equation corresponding to the determination of ν_0 then reads for $-1 < \lambda \leqslant 2$ (see (60))

$$(a \pm \nu_0)\nu_0^{\frac{1}{1+\lambda}} = Z, \tag{71}$$

where again $Z = Z(m, b, N^{(C)})$ is some unimportant function. Introducing again the integers p and q such that $1/(1+\lambda) = p/q$ and $x = \nu_0^{1/q}$, (71) becomes

$$(a \pm x^q)x^p = Z. \tag{72}$$

All the possibilities to choose p and q such that $p + q \leqslant 4$ are suitable. The list of solvable potentials is given below:

$$\lambda = -\tfrac{2}{3}, \ -\tfrac{1}{2}, \ 1, \ 2. \tag{73}$$

Among them we will study:

- $\lambda = 2$ because it corresponds to a potential $-a/r + br^2$ which is already studied with $P(r) = r^2$. This is the only potential that can be described with either $P(r) = r^2$ or $P(r) = -1/r$ and this property allows very fruitful comparisons.
- $\lambda = 1$ because it corresponds to the funnel potential (Coulomb + linear) which is widely used in hadron spectroscopy [4].

It could also be interesting to introduce potentials with $\lambda < -1$ but with the restriction that it is repulsive at the origin (for instance, Van der Walls forces or Lenhard–Jones type of potentials). In this case, the equation determining ν_0 is given by

$$a - \nu_0 = Z\nu_0^{\frac{1}{|\lambda|-1}}, \tag{74}$$

which can be transformed with $1/(|\lambda| - 1) = p/q$ and $x = \nu_0^{1/q}$ into

$$a - x^q = Zx^p. \tag{75}$$

The set of all solvable potentials is provided with the list below:

$$\lambda = -5, \ -4, \ -3, \ -\tfrac{5}{2}, \ -\tfrac{7}{3}, \ -2, \ -\tfrac{7}{4}, \ -\tfrac{5}{3}, \ -\tfrac{3}{2}, \ -\tfrac{4}{3}, \ -\tfrac{5}{4}. \tag{76}$$

In this list, we will just consider the case:

- $\lambda = -2$ because the corresponding potential, known as the Kratzer potential, exhibits its spectrum under an analytical form for all values of radial quantum number n and orbital quantum number l.

3.2. Kratzer potential

The Kratzer potential [7] is defined as

$$V(r) = \frac{a^2}{r^2} - \frac{2a}{r}. \tag{77}$$

It presents some interest as a benchmark since it is one of the rare potentials for which one knows an exact analytical expression of the energies valid for any n and l quantum numbers. Explicitly, one has

$$E(n, l) = -\frac{2ma^2}{[n + 1/2 + \sqrt{(l + 1/2)^2 + 2ma^2}]^2}. \tag{78}$$

Applying the AFM with $P(r) = -1/r$ to this potential leads to the following equation giving the energies as a function of the auxiliary field ν (in this section $N = N^{(C)} = n + l + 1$):

$$E(\nu) = -\frac{m}{2N^2}(2a - \nu)^2 - \frac{\nu^2}{4a^2}. \tag{79}$$

The minimization with respect to ν provides the value

$$\nu_0 = \frac{4ma^3}{2ma^2 + N^2} \tag{80}$$

which, inserted in (79), gives the desired result

$$E^{(K)}(n, l) = -\frac{2ma^2}{[2ma^2 + (n + l + 1)^2]}. \tag{81}$$

Let us remark that the approximate value $E^{(K)}$ presents the correct asymptotic behaviour for large n and for large l. Just to have an idea of the quality of this approximation, let us calculate the difference δ between the denominators of E and $E^{(K)}$. It is just a matter of simple algebra to find

$$\delta = (2n + 1)(l + 1/2) \left[1 - \sqrt{1 + \frac{2ma^2}{(l + 1/2)^2}} \right]. \tag{82}$$

Consequently, for small intensity and/or mass, $ma^2 \ll 1$, or for large angular momentum, $l \gg 1$, the approximate value tends to the exact one and we have more explicitly

$$\delta \to -(2n + 1)\frac{ma^2}{(l + 1/2)}. \tag{83}$$

This behaviour is easily understandable because, under those conditions, the contribution due to $1/r$ is predominant as compared to the contribution of $1/r^2$, and both expressions tend towards the same exact Coulomb result.

3.3. Quadratic + centrifugal potential

We consider now the potential (for an attractive centrifugal potential, not all values of b are relevant [9])

$$V(r) = ar^2 \pm \frac{b}{r^2}. \tag{84}$$

Incorporating the term $\pm b/r^2$ into the $l(l + 1)/r^2$ term already present in $\boldsymbol{p}^2$ allows us to get the exact eigenvalue using the same kind of arguments as those developed in the harmonic oscillator case [7]. Explicitly, we obtain

$$E(n, l) = \sqrt{\frac{a}{2m}} \left[2(2n + 1) + \sqrt{(2l + 1)^2 \pm 8mb} \right]. \tag{85}$$

Using the AFM with $P(r) = r^2$, the energies are given by (in this section $N = N^{(ho)} = 2n + l + 3/2$)

$$E(\nu) = \sqrt{\frac{2}{m}} N(a \mp \nu)^{1/2} \pm 2\sqrt{b}\nu^{1/2}. \tag{86}$$

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Setting $Y = 2mb/N^2$, the value ν_0 that minimizes this energy comes from a first degree equation and reads

$$\nu_0 = \frac{aY}{1 \pm Y}. \tag{87}$$

Substituting this value into the energy (86), one obtains a very simple expression for the approximate energy

$$E^{(qc)}(m, a, b; n, l) = 2\sqrt{\frac{a(N^2 \pm 2mb)}{2m}}. \tag{88}$$

This quantity and the corresponding exact one depend on three parameters m, a, b, but we know that the general scaling law properties allow us to write them in a more pleasant form

$$E(m, a, b; n, l) = \sqrt{\frac{a}{2m}}\epsilon(2mb; n, l), \tag{89}$$

where $\epsilon(\beta; n, l)$ is an eigenvalue of the reduced Schrödinger equation for which the Hamiltonian depends now on a single dimensionless parameter β

$$H = \boldsymbol{p}^2 + r^2 \pm \frac{\beta}{r^2}. \tag{90}$$

The exact eigenvalues of this Hamiltonian are given by

$$\epsilon(\beta; n, l) = 2(2n + 1) + \sqrt{(2l + 1)^2 \pm 4\beta}. \tag{91}$$

The approximate values immediately come from (88)

$$\epsilon^{(qc)}(\beta; n, l) = 2\sqrt{N^2 \pm \beta}. \tag{92}$$

One can check that the relative error between ϵ and $\epsilon^{(qc)}$ decreases as l^{-3} for a fixed value of n and decreases as n^{-1} for a fixed value of l. This behaviour is easily understandable because, for large values of the quantum numbers, the contribution due to r^2 is predominant as compared to the contribution of $1/r^2$ and both expressions tend towards the same exact harmonic oscillator result.

Let us assume that $\beta \ll 1$; a Taylor expansion truncated to first order leads to

$$\epsilon^{(qc)}(\beta; n, l) \approx 2N \pm \frac{\beta}{N}. \tag{93}$$

In particular for $\beta = 0$, one recovers the exact value $2N$, as expected. It is easy to check that this expression can also be obtained by perturbation theory, as expected from section 2.6.

3.4. Some reduced polynomial equations

The potentials that remain to be studied will need the solutions of cubic and quartic equations. In order to simplify as much as possible the formulae, we found interesting to put them in a form that makes the roots as simple as possible. The corresponding notations will be used in the following.

3.4.1. Cubic equation. It is interesting to work with a cubic equation of the form

$$x^3 + 3x - 2Y = 0. \tag{94}$$

There exists only one positive root given analytically by

$$F(Y) = \left[Y + \sqrt{1 + Y^2}\right]^{1/3} - \left[Y + \sqrt{1 + Y^2}\right]^{-1/3}. \tag{95}$$

When $Y \ll 1$, one has x close to 0 so that $x^3 \ll x$ and the behaviour of the root is simply

$$F(Y) \approx \frac{2Y}{3}, \qquad \text{if} \quad Y \ll 1. \tag{96}$$

When $Y \gg 1$, x is large so that $3x$ is negligible with respect to x^3 and we have the following behaviour:

$$F(Y) \approx (2Y)^{1/3}, \qquad \text{if} \quad Y \gg 1. \tag{97}$$

3.4.2. Quartic equation. The quartic equation which gives the most pleasant form for the roots is

$$4x^4 \pm 8x - 3Y = 0. \tag{98}$$

For each sign, there exists only one positive root given analytically by

$$G_\pm(Y) = \mp \tfrac{1}{2}\sqrt{V(Y)} + \tfrac{1}{2}\sqrt{4(V(Y))^{-1/2} - V(Y)}, \tag{99}$$

with

$$V(Y) = \left(2 + \sqrt{4 + Y^3}\right)^{1/3} - Y\left(2 + \sqrt{4 + Y^3}\right)^{-1/3}. \tag{100}$$

When $Y \ll 1$, one has $x^4 \ll x$ and the behaviour of the roots is simply

$$G_+(Y) \approx \frac{3Y}{8} \qquad \text{if} \quad Y \ll 1 \tag{101}$$

and

$$G_-(Y) \approx 2^{1/3} + \frac{Y}{8} \qquad \text{if} \quad Y \ll 1. \tag{102}$$

When $Y \gg 1$, x is large so that $8x$ is negligible with respect to $4x^4$ and we have the following behaviour:

$$G_\pm(Y) \approx (3Y/4)^{1/4} \qquad \text{if} \quad Y \gg 1. \tag{103}$$

3.5. Anharmonic potential

The potential under consideration reads

$$V(r) = ar^2 + 2br. \tag{104}$$

The interest for such a potential is discussed in section 3.1. Obviously, one must take $P(r) = r^2$ in the AFM and the energies to be considered are given by (in this section $N = N^{(\text{ho})} = 2n + l + 3/2$)

$$E(\nu) = \sqrt{\frac{2}{m}} N(a + \nu)^{1/2} + \frac{b^2}{\nu}. \tag{105}$$

Let us introduce the parameter

$$Y = \frac{8}{3} a \left(\frac{N^2}{mb^4}\right)^{1/3} \tag{106}$$

and the new variable

$$x = \frac{3Y}{8a} \nu. \tag{107}$$

The equation that leads to the minimization of the energy is then

$$4x^4 - 8x - 3Y = 0. \tag{108}$$

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This is the reduced quartic equation studied in the preceding section. The solution is given by $x_0(Y) = G_-(Y)$ (see (99)). Substituting this value into the energy (105) and making a little algebra leads to the desired approximate energy

$$E^{(an)}(m, a, b; n, l) = \frac{3b^2}{8a} Y \left(G_-^2(Y) + \frac{1}{G_-(Y)} \right),$$ (109)

with Y given by (106).

As in the previous case, this quantity and the corresponding exact one depend on three parameters m, a, b, but the general scaling law allows us to write them in terms of a reduced quantity depending on a single parameter β

$$E(m, a, b; n, l) = \sqrt{\frac{2a}{3m}} \epsilon \left(\frac{3b^2}{16} \sqrt{\frac{3m}{2a^3}}; n, l \right),$$ (110)

where $\epsilon(\beta; n, l)$ is an eigenvalue of the reduced Schrödinger equation for the Hamiltonian

$$H = \frac{p^2}{4} + 3r^2 + 8\sqrt{\beta} r.$$ (111)

The approximate value corresponding to this reduced equation follows from (109):

$$\epsilon^{(an)}(\beta; n, l) = 2\beta Y \left(G_-^2(Y) + \frac{1}{G_-(Y)} \right), \qquad Y = \left(\frac{N}{\beta} \right)^{2/3}.$$ (112)

The parameter β could also be associated with the quadratic potential, but this case less interesting is not considered here.

Let us assume that $\beta \ll 1$; a Taylor expansion truncated to first order leads to

$$\epsilon^{(an)}(\beta; n, l) \approx \sqrt{3} N + 4 \sqrt{\frac{2\beta N}{\sqrt{3}}}.$$ (113)

In particular for $\beta = 0$, one recovers the exact value $\sqrt{3} N$, as should be. This result also comes from the perturbation theory.

The limit $\beta \to \infty$ is not physically relevant, but it is interesting to consider it in order to check the formula. In this limit, we find

$$\epsilon^{(an)}(\beta; n, l) = 3(4\beta N^2)^{1/3} + O(\beta^{-1/3}).$$ (114)

The dominant term is the result expected for a pure linear potential, as given by (10).

3.6. Quadratic + Coulomb potential

In this section, we study the quadratic + Coulomb potential defined as

$$V(r) = ar^2 - \frac{b}{r}.$$ (115)

The interest for such a potential is discussed in section 3.1. Let us illustrate the AFM method with the option $P(r) = r^2$ so that $N = N^{(ho)} = 2n + l + 3/2$. The energies depending on the auxiliary field ν are given in this case by

$$E(\nu) = \sqrt{\frac{2}{m}} N(a + \nu)^{1/2} - 3 \left(\frac{b^2\nu}{4} \right)^{1/3}.$$ (116)

Let us introduce the parameter

$$Y = \frac{8N^2}{3m} \left(\frac{4a}{b^4} \right)^{1/3}$$ (117)

and the new variable

$$x = \left(\frac{2a}{v}\right)^{1/3}. \tag{118}$$

The equation that leads to the minimization of the energy is then

$$4x^4 + 8x - 3Y = 0. \tag{119}$$

This reduced quartic equation was studied previously. The solution is given by $x_0(Y) = G_+(Y)$ (see (99)). Substituting this value into the energy (116), one is led, after some manipulations, to the desired approximate energy

$$E^{(qC)}(m, a, b; n, l) = \frac{3}{4}\left(\frac{ab^2}{2}\right)^{1/3}\left[\frac{Y}{G_+^2(Y)} - \frac{4}{G_+(Y)}\right], \tag{120}$$

with Y given by (117).

One can check that, starting with $P(r) = -1/r$ and performing the same kind of algebra, the approximate energy is given by exactly the same equation as (120), but this time with the Y parameter given by (117) in which $N^{(ho)}$ is replaced by $N^{(C)}$. This property is a very nice check of the general prescription demonstrated in section 2.3.

The quantity (120) and the corresponding exact one depend again on three parameters m, a, b, but the general scaling law allows us to write them in terms of a reduced quantity depending on a single dimensionless parameter β. One can imagine two formulations depending on whether β is part of the quadratic contribution or of the Coulomb contribution:

$$E(m, a, b; n, l) = 4\sqrt{\frac{2a}{3m}}\,\epsilon\left(\frac{1}{4}\left(\frac{54m^3b^4}{a}\right)^{1/6}; n, l\right), \tag{121}$$

$$E(m, a, b; n, l) = \frac{3mb^2}{16}\,\eta\left(4\left(\frac{a}{54m^3b^4}\right)^{1/6}; n, l\right). \tag{122}$$

The ϵ and η energies are the eigenvalues of the reduced Schrödinger equations for the respective Hamiltonians H_ϵ and H_η:

$$H_\epsilon = \frac{3p^2}{16} + \frac{r^2}{4} - \frac{\beta^{3/2}}{r}, \tag{123}$$

$$H_\eta = \frac{3p^2}{16} - \frac{\sqrt{2}}{r} + \beta'^6 r^2. \tag{124}$$

The approximate values corresponding to these reduced Hamiltonians follow from (120):

$$\epsilon^{(qC)}(\beta; n, l) = \frac{3\beta}{8}\left[\frac{Y}{G_+^2(Y)} - \frac{4}{G_+(Y)}\right], \qquad Y = \left(\frac{N}{\beta}\right)^2, \tag{125}$$

$$\eta^{(qC)}(\beta'; n, l) = \frac{3\beta'^2}{4}\left[\frac{Y}{G_+^2(Y)} - \frac{4}{G_+(Y)}\right], \qquad Y = \left(N\beta'\right)^2. \tag{126}$$

Let us assume that, $\beta \ll 1$, that is $V(r) \ll P(r)$:

- A Taylor expansion truncated to first order for the formulation based on the ϵ form leads to

$$\epsilon^{(qC)}(\beta; n, l) \approx \frac{\sqrt{3}}{4}N - \sqrt{\frac{2\beta^3}{N\sqrt{3}}}. \tag{127}$$

In particular for $\beta = 0$, one recovers the exact value $\sqrt{3}N/4$ if $N = N^{(ho)}$.

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- The Taylor expansion for the formulation based on the η form gives

$$\eta^{(qC)}(\beta'; n, l) \approx -\frac{8}{3N^2} + \frac{9\beta'^6 N^4}{128}. \tag{128}$$

In particular for $\beta' = 0$, one recovers the exact value $-8/(3N^2)$ if $N = N^{(C)}$.

In all cases, the approximate formulae resulting from AFM agree with the result of perturbation theory.

The limit $\beta \to \infty$ is interesting to consider in order to check the formulae:

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$$\epsilon^{(qC)}(\beta; n, l) = -\frac{4\beta^3}{3N^2} + O(\beta^{-2}). \tag{129}$$

If $N = N^{(C)}$, the dominant term is the exact result for a Coulomb potential.

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$$\eta^{(qC)}(\beta'; n, l) = \frac{\sqrt{3}}{2}\beta'^3 N + O(\beta'^{3/2}). \tag{130}$$

If $N = N^{(ho)}$, the dominant term is the exact result for a quadratic potential.

Let us emphasize the point that both (123) and (124) can be related to the scaling laws developed in section 2.4. As a consequence, it can be shown that both the exact eigenvalues and the AFM approximate ones fulfil the relation

$$\epsilon(\beta; n, l) = \frac{\beta^3}{2}\eta(1/\beta; n, l). \tag{131}$$

It is worth mentioning that analytical solutions of the Schrödinger equation with quadratic plus Coulomb potentials are presented in [10], within the framework of a two-dimensional system of two-interacting electrons $(-1/r)$ in a confining magnetic field (r^2). Closed-form solutions are found for particular values of the magnetic field and spatial confinement length. It is argued that a generalization to a three-dimensional space is possible. But, as the corresponding formulae are not given explicitly, a comparison with our results is not available. More generally, the Schrödinger equation with a $-a/r + br + cr^2$ potential (particular cases are studied in sections 3.6 and 3.7) is directly linked with the biconfluent Heun's equation [11].

3.7. Funnel potential

In this section, we are concerned with the funnel potential defined as

$$V(r) = ar - \frac{b}{r}. \tag{132}$$

This potential is particularly important and its interest is discussed in section 3.1. Let us recall that it is widely used in hadronic spectroscopy and corresponds to a linear confinement coupled to a short-range Coulomb contribution [4]. Finding approximate analytical values for the energies corresponding to this potential is thus a very interesting question. To our knowledge such formulae are not proposed in the literature.

Naturally, one must apply the AFM method with $P(r) = -1/r$ so that $N = N^{(C)} = n + l + 1$. The energies depending on the auxiliary field ν can be calculated following the prescription detailed in section 2.5. Explicitly, one finds

$$E(\nu) = -\frac{m(b + \nu)^2}{2N^2} + 2\sqrt{a}\nu^{1/2}. \tag{133}$$

Let us introduce the parameter

$$Y = \frac{3}{2} N^2 \sqrt{\frac{3a}{m^2 b^3}} \tag{134}$$

and the new variable

$$x = \sqrt{\frac{3v}{b}}. \tag{135}$$

The equation that must be solved to minimize the energy is

$$x^3 + 3x - 2Y = 0. \tag{136}$$

This reduced cubic equation was studied previously. The solution is given by $x_0(Y) = F(Y)$ (see (95)). Inserting this value into the energy (133), one finds the expression of the approximate energy

$$E^{(\mathrm{f})}(m, a, b; n, l) = \sqrt{3ab} \left[\frac{Y}{F^2(Y)} - \frac{2}{F(Y)} \right], \tag{137}$$

with Y given by (134). Let us mention that another form of this equation can be found thanks to the following relation:

$$\frac{Y}{F^2(Y)} - \frac{2}{F(Y)} = \sinh\theta - \frac{1}{4\sinh\theta}, \tag{138}$$

with the change of variables $Y = \sinh(3\theta)$.

The quantity (137) and the corresponding exact one depend again on three parameters m, a, b, but the general scaling law allows us to write them in terms of a reduced quantity depending on a single dimensionless parameter β. In hadronic physics, the dominant interaction between a quark and an antiquark is a confining linear potential [4]. So, although the linear potential has no analytical exact solution for all values of the quantum numbers, it is also interesting to consider two formulations depending on whether β is part of the linear contribution or of the Coulomb contribution:

$$E(m, a, b; n, l) = 3 \left(\frac{a^2}{2m} \right)^{1/3} \epsilon \left(\left(\frac{4m^2 b^3}{27a} \right)^{1/4}; n, l \right), \tag{139}$$

$$E(m, a, b; n, l) = \frac{2mb^2}{3^{5/3}} \eta \left(\left(\frac{27a}{4m^2 b^3} \right)^{1/4}; n, l \right). \tag{140}$$

The ϵ and η energies are the eigenvalues of the reduced Schrödinger equations for the respective Hamiltonians H_ϵ and H_η:

$$H_\epsilon = \frac{p^2}{3} + \frac{r}{3} - \frac{\beta^{4/3}}{r}, \tag{141}$$

$$H_\eta = \frac{p^2}{3} - \frac{3^{1/3}}{r} + \beta'^4 r. \tag{142}$$

The approximate values corresponding to these reduced Hamiltonian follow from (137):

$$\epsilon^{(\mathrm{f})}(\beta; n, l) = \beta^{2/3} \left[\frac{Y}{F^2(Y)} - \frac{2}{F(Y)} \right], \qquad Y = \left(\frac{N}{\beta} \right)^2, \tag{143}$$

$$\eta^{(\mathrm{f})}(\beta'; n, l) = 3^{2/3} \beta'^2 \left[\frac{Y}{F^2(Y)} - \frac{2}{F(Y)} \right], \qquad Y = (N\beta')^2. \tag{144}$$

Let us assume that $\beta \ll 1$, that is $V(r) \ll P(r)$:

- The Taylor expansion for the formulation based on the ϵ form gives

$$\epsilon^{(f)}(\beta; n, l) \approx \frac{N^{2/3}}{2^{2/3}} - \left(\frac{\beta^4}{2N^2}\right)^{1/3}. \tag{145}$$

In particular for $\beta = 0$, one recovers the value expected for a pure linear potential, as given by (10).

- The Taylor expansion for the formulation based on the η form gives

$$\eta^{(f)}(\beta'; n, l) \approx -\frac{3^{5/3}}{4N^2} + \frac{2N^2\beta'^4}{3^{4/3}}. \tag{146}$$

In particular for $\beta' = 0$, one recovers the exact value $-3^{5/3}/(4N^2)$.

In all cases, the approximate formulae resulting from AFM agree with the result of perturbation theory.

The limit $\beta \to \infty$ is interesting to consider in order to check the formulae:

-

$$\epsilon^{(f)}(\beta; n, l) = -\frac{3\beta^{8/3}}{4N^2} + O(\beta^{-4/3}). \tag{147}$$

The dominant term is the exact result for a Coulomb potential.

-

$$\eta^{(f)}(\beta'; n, l) = \left(\tfrac{3}{2}\beta'^4 N\right)^{2/3} + O(\beta'^{4/3}). \tag{148}$$

The dominant term is the result expected for a pure linear potential, as given by (10).

Here again, both (141) and (142) can be related to the scaling laws. As a consequence, it can be shown that both exact eigenvalues and AFM approximate ones fulfil the relation

$$\epsilon(\beta; n, l) = \frac{\beta^{8/3}}{3^{2/3}}\eta(1/\beta; n, l). \tag{149}$$

4. Comparison with numerical results

In the previous section, approximate analytical forms for eigenvalues of several Hamiltonians were found. The formulae depend on the quantum numbers n and l through a factor N. This number could be taken as $N^{(ho)}$, $N^{(C)}$ or even N_λ if the potential $P(r)$ chosen is the power-law potential $P^{(\lambda)}(r)$ (see section 2.3). If we look, for instance, at Hamiltonian (123), it is clear that it reduces to a harmonic oscillator when $\beta = 0$. In this case, the choice $N = N^{(ho)}$ gives the exact result. When $\beta \to \infty$, the Coulomb part dominates and the choice $N = N^{(C)}$ is expected to yield the exact result asymptotically.

All dimensionless Hamiltonians considered above depend on a parameter β. The variation of the eigenvalues being smooth for the variation of β, we can assume that the number N, giving the optimal values for a selected set of these eigenvalues, is also a smooth function of β. From considerations above, the functional form

$$N(\beta) = b(\beta)n + l + c(\beta) \tag{150}$$

seems reasonable. If, in the limits $\beta \to 0$ and $\beta \to \infty$, the Hamiltonian reduces to a known form, the values $N(0)$ and $N(\infty)$ can be computed. But for finite values of β, we cannot predict the correct behaviour. It is then necessary to focus our attention on numerical solutions.
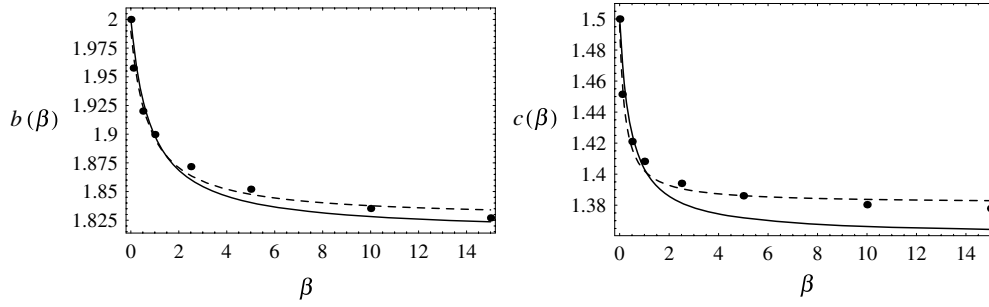
Figure 1. Best values of the coefficients $b(\beta)$ and $c(\beta)$ to parameterize the eigenvalues of Hamiltonian (153): numerical fit with (151) (dots); functions (154) with set 1 of parameters from table 1 (solid line); same with set 2 (dashed line).

Very accurate eigenvalues $\epsilon_{\mathrm{num}}(\beta; n, \ell)$ for Hamiltonians defined above can be obtained numerically with the Lagrange mesh method [12]. It is very accurate and easy to implement. In order to find the best possible values for coefficients $d(\beta)$ (d stands for b or c), we will use the following measure:

$$\chi(\beta) = \frac{1}{16} \sum_{n=0}^{3} \sum_{\ell=0}^{3} (\epsilon_{\mathrm{num}}(\beta; n, \ell) - \epsilon_{\mathrm{app}}(\beta; n, \ell))^2, \tag{151}$$

where $\epsilon_{\mathrm{app}}(\beta; n, \ell)$ are values obtained from our approximate formulae. Other choices are possible but we find this one very convenient. The analytical form ϵ_{app} depends on $N(\beta)$ which depends on coefficients d. For each value of β, optimal values for the d coefficients, $d_{\mathrm{min}}(\beta)$, can be determined by minimizing $\chi(\beta)$. Then, with a set $\{d_{\mathrm{min}}(\beta)\}$ for a given set $\{\beta\}$, a functional form $d_{\mathrm{fit}}(\beta)$ can be fitted with the following measure:

$$\chi(d) = \sum_{\{\beta\}} (d_{\mathrm{min}}(\beta) - d_{\mathrm{fit}}(\beta))^2. \tag{152}$$

Again, other choices are possible but we find this one very convenient. We will now try to determine the best form of coefficients $d(\beta)$ for some of the potentials studied above.

4.1. Improvement for anharmonic potential

The dimensionless Hamiltonian proposed here for the anharmonic potential is

$$H = \frac{p^2}{4} + 3r^2 + 8\sqrt{\beta}r. \tag{153}$$

Approximate eigenvalues are given by (112). With (150), the exact result will be obtained for $b(0) = 2$ and $c(0) = 3/2$. When $\beta \to \infty$, the linear part dominates and, from results obtained in SSB, we could expect that $b(\infty) \approx \pi/\sqrt{3} \approx 1.814$ and $c(\infty) \approx \sqrt{3}\pi/4 \approx 1.360$. By minimizing our measure $\chi(\beta)$, we found the optimal values of $b(\beta)$ and $c(\beta)$ for several finite values of β. The results are plotted with dots in figure 1. One can clearly see the smooth transition between two domains for zero and infinite values of β. This corresponds to the transition between two potentials not too different, a linear one and a quadratic one.

We tried to fit the numerical points with various functions and found that the best result is obtained for sections of hyperbola, both for $b(\beta)$ and $c(\beta)$. The parameterization retained is

$$b(\beta) = \frac{p_1\beta + p_2}{\beta + p_3}, \qquad c(\beta) = \frac{q_1\beta + q_2}{\beta + q_3}. \tag{154}$$

Table 1. Values of parameters p_i and q_i for (154). Fixed parameters are marked by a *.

	p_1	p_2	p_3	q_1	q_2	q_3
Set 1	$\pi/\sqrt{3}*$	$2p_3*$	0.835	$\sqrt{3}\pi/4*$	$3q_3/2*$	0.445
Set 2	1.826	1.485	0.747	1.381	0.333	0.222

Table 2. Values of $\chi(\beta)$ for eigenvalues of Hamiltonian (153) as a function of β, for various parameterizations of $N = b(\beta)n + l + c(\beta)$ (see table 1). $N^{(ho)}$ and set 1 give the exact result for $\beta = 0$.

β	$N^{(ho)}$	Set 1	Set 2
0.1	5.9×10^{-2}	6.2×10^{-3}	4.0×10^{-3}
1	0.46	2.7×10^{-3}	1.9×10^{-3}
10	2.8	1.8×10^{-2}	5.4×10^{-3}

With this choice, $b(0) = p_2/p_3$, $b(\infty) = p_1$, $c(0) = q_2/q_3$ and $c(\infty) = q_1$. Two different fits are presented in table 1. For set 1, only one parameter is free for each coefficient and the following constraints are imposed: $b(0) = 2$, $b(\infty) = \pi/\sqrt{3}$, $c(0) = 3/2$, $c(\infty) = \sqrt{3}\pi/4$. For set 2, all parameters are free; the results found by minimization give: $b(0) \approx 1.990$ close to 2, $b(\infty) = 1.826$ close to 1.814 ($\pi/\sqrt{3}$), $c(0) \approx 1.496$ close to 1.5, $c(\infty) = 1.381$ close to 1.360 ($\sqrt{3}\pi/4$).

The quality of the fits can be appraised by examining the values of $\chi(\beta)$ shown in table 2. It is clear that allowing a β-dependence for the coefficients b and c improves greatly the approximate eigenvalues. Let us remark that the fit with three parameters is only slightly better than the fit with only one parameter, and that $\chi(\beta)$ increases with β for the choice $N = N^{(ho)}$, as expected since the potential deviates more and more from a pure quadratic one.

4.2. Improvement for quadratic + Coulomb potential

To study the case of the quadratic + Coulomb potential, we choose the following dimensionless Hamiltonian:

$$H = \frac{3\boldsymbol{p}^2}{16} + \frac{r^2}{4} - \frac{\beta^{3/2}}{r}. \tag{155}$$

Approximate eigenvalues are given by (125). The exact result will be obtained for $b(0) = 2$ and $c(0) = 3/2$. When $\beta \to \infty$, the Coulomb part dominates and we could expect that $b(\infty) = 1$ and $c(\infty) = 1$. By minimizing our measure $\chi(\beta)$, we found the optimal values of $b(\beta)$ and $c(\beta)$ for several finite values of β. The results are plotted with dots in figure 2. One can clearly see the smooth transition between the harmonic oscillator region near zero and the asymptotic Coulomb region. In this case, the transition occurs between two very different potentials: r^2 and $-1/r$. It is clear that the asymptotic region is more rapidly reached for $c(\beta)$ than for $b(\beta)$. When the quantum numbers n and l increase, the size of the eigenfunctions grows and the states become less sensitive to the Coulomb part. So, we can understand that the influence of β on the coefficient $b(\beta)$ is less significant than on $c(\beta)$.

We tried to fit the numerical points with various functions and found that the best result is obtained for exponential functions with a cubic argument, both for $b(\beta)$ and $c(\beta)$. The parameterization retained is

$$b(\beta) = 1 + p_1 \exp(-p_2(\beta - p_3)^3), \qquad c(\beta) = 1 + q_1 \exp(-q_2(\beta - q_3)^3). \tag{156}$$
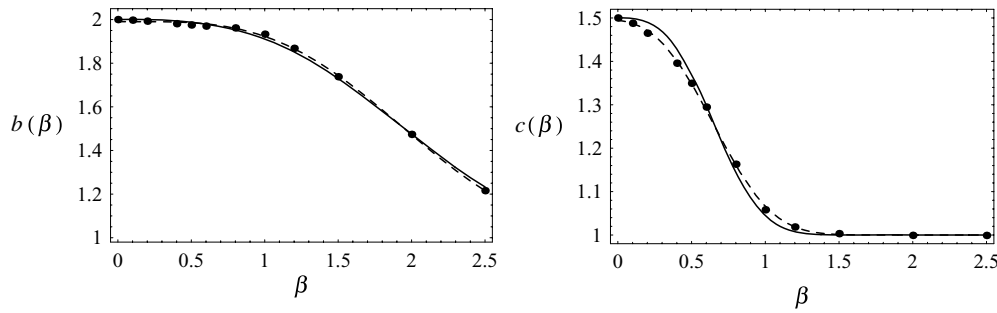
Figure 2. Best values of the coefficients $b(\beta)$ and $c(\beta)$ to parameterize the eigenvalues of Hamiltonian (155): numerical fit with (151) (dots); functions (156) with set 1 of parameters from table 3 (solid line); same with set 2 (dashed line).

Table 3. Values of parameters p_i and q_i for (156). Fixed parameters are marked by a *.

	p_1	p_2	p_3	q_1	q_2	q_3
Set 1	1*	0.093	0*	1/2*	2.414	0*
Set 2	0.990	0.119	0.161	0.496	1.373	−0.136

Table 4. Values of $\chi(\beta)$ for eigenvalues of Hamiltonian (155) as a function of β, for various parameterizations of $N = b(\beta)n + l + c(\beta)$ (see table 3). $N^{(\text{ho})}$ gives the exact result for $\beta = 0$, $N^{(\text{C})}$ gives the exact result for $\beta = \infty$, set 1 gives the exact result for both $\beta = 0$ and $\beta = \infty$, and set 2 gives the exact result for $\beta = \infty$.

β	$N^{(\text{ho})}$	$N^{(\text{C})}$	Set 1	Set 2
0.5	9.6×10^{-3}	0.90	2.6×10^{-3}	2.3×10^{-3}
1	0.11	0.75	1.9×10^{-2}	1.8×10^{-2}
2	3.1	0.45	0.12	0.12

With this choice, $b(\infty) = 1$ and $c(\infty) = 1$. Two different fits are presented in table 3. For set 1, only one parameter is free for each coefficient and the following constraints are imposed: $b(0) = 2$ and $c(0) = 3/2$. For set 2, all parameters are free; the results found by minimization give: $b(0) \approx 1.991$ close to 2 and $c(0) \approx 1.495$ close to 1.5.

The quality of the fits can be appraised by examining the values of $\chi(\beta)$ shown in table 4. Again, allowing a β-dependence for the coefficients b and c improves greatly the approximate eigenvalues. The fits, with one parameter and three parameters, give nearly the same result. $\chi(\beta)$ increases with β for the choice $N = N^{(\text{ho})}$, as expected since the potential deviates more and more from a pure quadratic one. In contrast, $\chi(\beta)$ decreases with β for the choice $N = N^{(\text{C})}$, as expected since the potential comes closer to a pure Coulomb one. The values of $\chi(\beta)$ are large because the considered values of β are not in the asymptotic region of $b(\beta)$, as one can see in figure 2. It has been checked that $\chi(\beta) \to 0$ also for sets 1 and 2 for large values of β.

4.3. Improvement for funnel potential

For the case of the funnel potential, we choose the following dimensionless Hamiltonian:

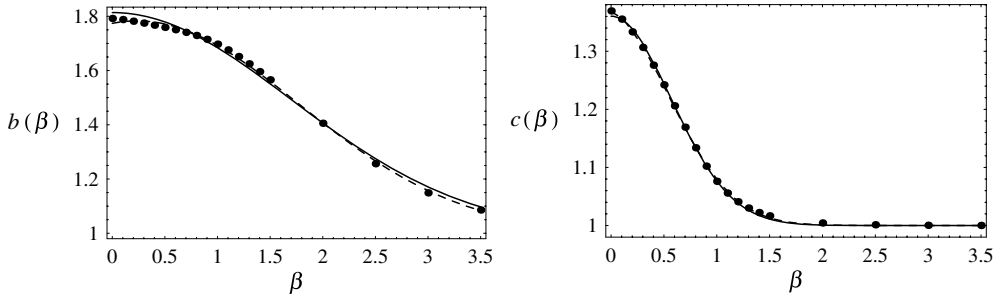$$H = \frac{p^2}{3} + \frac{r}{3} - \frac{\beta^{4/3}}{r}. \tag{157}$$

Figure 3. Best values of the coefficients $b(\beta)$ and $c(\beta)$ to parameterize the eigenvalues of Hamiltonian (157): numerical fit with (151) (dots); functions (158) with set 1 of parameters from table 5 (solid line); same with set 2 (dashed line).

Table 5. Values of parameters p_i and q_i for (158). Fixed parameters are marked by a *.

	p_1	p_2	p_3	q_1	q_2	q_3
Set 1	$\frac{\pi}{\sqrt{3}} - 1^*$	0.416	0^*	$\frac{\sqrt{3}\pi}{4} - 1^*$	1.245	0^*
Set 2	0.783	0.459	0.237	0.369	1.168	-0.062

Approximate eigenvalues are given by (143). This choice is motived by three reasons:

- for hadronic problems [4], physical values of β varies from 0 to about 1.5;
- a comparison is possible between Hamiltonians (157) and (155) which differ by the 'confinement' part;
- in contrast to previous cases, $\beta = 0$ does not correspond to a full analytical case.

For $\beta = 0$, the potential is pure linear one. So, from the study performed in SSB, we can expect that $b(0) = \pi/\sqrt{3} \approx 1.814$ and $c(0) = \sqrt{3}\pi/4 \approx 1.360$ is a good choice. When $\beta \to \infty$, the Coulomb part dominates and we could expect that $b(\infty) = 1$ and $c(\infty) = 1$. By minimizing our measure $\chi(\beta)$, we found the optimal values of $b(\beta)$ and $c(\beta)$ for several finite values of β. The results are plotted with dots in figure 3. One can see a smooth transition similar to the previous case. Again, for probably the same reason, the asymptotic region is more rapidly reached for $c(\beta)$ than for $b(\beta)$.

We tried to fit the numerical points with various functions and found that the best result is obtained for a Gaussian function both for $b(\beta)$ and $c(\beta)$. The parameterization retained is

$$b(\beta) = 1 + p_1 \exp(-p_2^2(\beta - p_3)^2), \qquad c(\beta) = 1 + q_1 \exp(-q_2^2(\beta - q_3)^2). \qquad (158)$$

Let us note the difference with the previous case, for which the argument of the exponential was cubic in β and not quadratic. With this choice, $b(\infty) = 1$ and $c(\infty) = 1$. Two different fits are presented in table 5. For set 1, only one parameter is free for each coefficient and the following constraints are imposed: $b(0) = \pi/\sqrt{3}$ and $c(0) = \sqrt{3}\pi/4$. For set 2, all parameters are free; the results found by minimization give: $b(0) \approx 1.774$ close to $\pi/\sqrt{3}$ (1.814) and $c(0) \approx 1.367$ close to $\sqrt{3}\pi/4$ (1.360).

The quality of the fits can be appraised by examining the values of $\chi(\beta)$ shown in table 6 and the values of approximate results compared with exact ones presented in table 7. As in the two previous cases, allowing a β-dependence for the coefficients b and c improves greatly the approximate eigenvalues. The fits with one parameter and three parameters also give nearly the same results and the behaviour of $\chi(\beta)$ for this case and the previous one are very similar.

Table 6. Values of $\chi(\beta)$ for eigenvalues of Hamiltonian (157) as a function of β, for various parameterizations of $N = b(\beta)n + l + c(\beta)$ (see table 5). $N^{(C)}$, set 1 and set 3 give the exact result for $\beta = \infty$.

β	$N^{(C)}$	Set 1	Set 2
0.5	0.17	4.2×10^{-4}	3.6×10^{-4}
1	0.15	2.9×10^{-3}	2.9×10^{-3}
2	0.11	1.7×10^{-2}	1.7×10^{-2}

Table 7. Eigenvalues $\epsilon(\beta_0, n, l)$ of Hamiltonian (157) with $\beta_0 = 0.5$, for some sets (n, l). First line: $\epsilon_{\mathrm{num}}(\beta_0; n, \ell)$ from numerical integration; second line: $\epsilon^{(f)}(\beta_0; n, l)$ given by (143) with $N(\beta)$ defined by set 1; third line: $\epsilon^{(f)}(\beta_0; n, l)$ given by (143) with $N(\beta) = N^{(C)}$.

l	$\epsilon(\beta_0; 0, l)$	$\epsilon(\beta_0; 1, l)$	$\epsilon(\beta_0; 2, l)$	$\epsilon(\beta_0; 3, l)$
0	0.397 11	1.117 14	1.645 58	2.096 28
	0.427 79	1.162 23	1.680 99	2.122 05
	0.268 27	0.791 05	1.154 40	1.459 87
1	0.905 98	1.459 55	1.925 80	2.341 67
	0.887 94	1.466 73	1.935 64	2.349 11
	0.791 05	1.154 40	1.459 87	1.732 69
2	1.257 49	1.742 47	2.171 33	2.562 88
	1.233 07	1.738 92	2.173 23	2.565 06
	1.1544	1.459 87	1.732 69	1.983 58
3	1.554 57	1.997 27	2.399 17	2.771 68
	1.529 08	1.989 37	2.397 64	2.771 83
	1.459 87	1.732 69	1.983 58	2.218 33

4.4. General considerations

From the results of this section, it is clear that a good choice for the function $N(\beta)$ can greatly improve the accuracy of the energy formulae. Unfortunately, the best functional form cannot be theoretically predicted. In the cases studied in this paper, the same form can be given to both coefficients $b(\beta)$ and $c(\beta)$ for a given Hamiltonian, but with different parameters. The behaviour for $\beta = 0$ and $\beta \to \infty$ can sometimes be exactly computed.

Obviously, the parameters for coefficients $b(\beta)$ and $c(\beta)$ depend on the points used for the fit but also on the particular choice of functions (151) and (152). Other definitions—relative error instead of absolute error, different chosen quantum numbers or different chosen values of β in summations—would have given other numbers slightly different. It is worth noting that, for each case studied, values of $\chi(\beta)$ obtained directly with coefficients $d_{\mathrm{min}}(\beta)$ and $d_{\mathrm{fit}}(\beta)$ for the set 2 are generally very close.

5. Conclusions

The AFM was proposed in SSB as a tool to compute approximate analytical solutions of the Schrödinger equation and then applied to the case of power-law radial potentials. The basic idea underlying this method is to replace an arbitrary potential $V(r)$, for which no analytical spectrum is known, by an expression of the type $\nu P(r) + g(\nu)$, $P(r)$ being a potential for which analytical eigenenergies can be found, ν the auxiliary field and $g(\nu)$ a well-defined function

of this extra parameter. This auxiliary field is such that its elimination as an operator leads to the original Hamiltonian. If ν is seen as a number however, analytical eigenenergies and eigenstates can be found, and the auxiliary field is eventually eliminated by a minimization on the eigenenergies. The approximation in the AFM comes thus from the use of the auxiliary field as a number rather than an operator.

In the present work, we have further investigated the AFM and obtained results that fall mainly in two categories: general properties of the AFM and analytical resolution of the Schrödinger equation with potentials of the form $ar^\lambda \pm br^\eta$. Let us first summarize the general properties of the AFM that we have proved:

- The analytical expressions that are obtained for the eigenenergies by using the AFM preserve the general scaling laws (30) in the case where $P(r)$ is homogeneous. Thanks to this feature, the number of relevant parameters in the Hamiltonian can be seriously reduced in order to simplify the problem.
- Let $P_1(r)$ and $P_2(r)$ be two power-law potentials whose eigenenergies respectively read $E_1(N_1)$ and $E_2(N_2)$, where N_1 and N_2 are terms containing the radial and orbital quantum numbers. Then, if we apply the AFM to find the eigenenergies of potential $V(r)$ with both $P_1(r)$ and $P_2(r)$, the final results will have the same functional form, depending either on N_1 or on N_2 following the case.
- Perturbation theory can be reformulated within the AFM. If σ is a small parameter and if $V(r) = \sigma v(r)$ is a potential that can be treated in perturbation, then $\langle v(r) \rangle$ can be equivalently replaced by $v(J(\nu_0))$ with $J(\nu_0)$ the average point defined by (50), at the first order in σ. Such a property could be useful in some cases since no integration is needed with the AFM.

Finally, it is worth summing up the results that we obtained by solving the Schrödinger equation with potentials of the form $ar^\lambda \pm br^\eta$:

- Analytical formulae have been found for several potentials that are relevant in various domain of physics: anharmonic, quadratic plus Coulomb and funnel. To our knowledge, it is the first time that such formulae are found. A comparison with the Kratzer and the quadratic plus centrifugal potentials which are analytically solvable is also performed.
- By using the scaling laws, it appears that only one dimensionless parameter, denoted as β in this work, 'controls' the features of the various spectra. The approximate formulae for eigenenergies can be a complicated function of β and of a number N containing the radial and orbital quantum numbers. This number is determined by the potential $P(r)$ chosen and is *a priori* independent of β. Nevertheless, a drastic improvement of the formulae we computed can be obtained by replacing this term N by a function of the form $b(\beta)n + \ell + c(\beta)$, where $b(\beta)$ and $c(\beta)$ have to be fitted on the eigenenergies coming from a numerical resolution of the Schrödinger equation. This has finally led us to very accurate formulae for all the various potentials we studied.

As an outlook, we mention that more complicated potentials could be studied with the AFM. In particular, we plan to apply this method to potentials involving an exponential, like the Yukawa potential for example. Such a work is in progress.

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